

# THE MULTITUDE OF TRIAD SYSTEMS ON 31 LETTERS\*

BY

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Triad systems, or triple systems, apparently different, on 15 letters have long been known; and as early as 1859 Reiss established the existence of at least one triad system for every number of letters,  $t$ , of the form  $6k + 1$  or  $6k + 3$ . E. H. Moore in 1893 was the first to prove† that when  $t$  is above 13 there are at least two non-equivalent systems. This he proved by showing a difference in the groups that leave the two systems respectively invariant. He found reason also for the conjecture that a particular class of such systems, those whose group contains a cyclic substitution of order  $t$ , increase in number rapidly with increasing  $t$ . For the number  $t = 13$ , which Moore had left in doubt, Zulauf then proved\* that there are at least two distinct systems; and it was demonstrated finally by De Pasquale‡ (later also by Brunel§) that for  $t = 13$  there are only those two systems. Zulauf's proof was by the use of the groups of the systems.

While two systems belonging to different groups are obviously not equivalent—that is, they cannot be transformed into one another by any substitution on the letters, Miss Cummings§ has found that the converse is not true, and has given examples of two systems having the same group, which yet are not equivalent. She has applied a new method of comparison, that of sequences and indices, and has constructed for  $t = 15$  so many new systems that their number is now known to be not less than 24. I have now been able, by a reasonably short method, to make for  $t = 15$  an exhaustive catalogue of all triad systems whose group is of order at least 2. By this means there are disclosed a considerable number additional, of a special kind previously

\* Presented to the Society, September 8, 1914.

† E. H. Moore, *Concerning triple systems*, *Mathematische Annalen*, vol. 43 (1893), pp. 271–285.

\* K. Zulauf, *Ueber Tripelsysteme von 13 Elementen*. Dissertation, Marburg, 1897.

† *Sui sistemi ternari di 13 elementi*. *Rendiconti R. Istituto Lombardo*, ser. 2, vol. 32 (1899).

‡ *Sur les deux systèmes de triades de treize éléments*. *Journal de Mathématiques*, ser. 5, vol. 7 (1901). See also an article cited later, by F. N. Cole, containing a complete demonstration.

§ *On a method of comparison for triple-systems*. *These Transactions*, vol. 15 (1914), p. 311.

known only in a single example. The method and results of this study will soon be laid before this Society; but my present purpose is to apply some of the results to the next question in order.

As soon as we possess one or more triad systems on 15 letters, systems on 31 can be constructed. With the aid of a new theorem, and a novel property of the arrangement of the 120 pairs on 15 letters in an "array" of 15 columns of 8, I succeed in constructing a simple kind of system on 31 letters and proving that a very great number of such systems are mutually non-equivalent. How many kinds there are, and even the precise number of this most simple kind, I do not determine. But we do see that their number is so overwhelmingly great that further study cannot be that of details and individual systems, but must necessarily be a search for general characters, relations, and classification.

From the small number of triad systems hitherto written out explicitly, and the slow rate of their increase up to  $t = 15$ , the totality of triad-systems has been considered to be comparable to that of polyhedra. From present indications, however, their field appears to be far more extensive. Its fertility is yet to be determined.

### 1. THE ODD AND EVEN STRUCTURE

Certain triad systems admit a separation of their letters into two sets, those of the one set appearing by ones or threes in all the triads, while those of the other set appear (consequently) either two together or not at all. Those sets, when the separation is possible, we term respectively an odd set and an even set. It is easily seen that the even set must contain one more letter than the odd set; and that the triads containing the odd set exclusively must constitute a closed system, a complete triad system contained in the larger one. Every triad system on 15 letters that has been described up to the present time exhibits this odd and even structure, i. e., contains a triad system on 7 letters (a  $\Delta_7$ ) and an even set of 8; with one exception, namely the cyclic  $\Delta_{15}$  of Heffter. We shall call the odd set the *head*, and so classify Heffter's cyclic system as *headless*. Using the usual notation we say that a  $\Delta_{2n+1}$  exhibiting the odd and even structure has for head a  $\Delta_n$ . If, conversely, a  $\Delta_{2n+1}$  contains a  $\Delta_n$ , it has odd and even structure with that  $\Delta_n$  for odd set. Often a  $\Delta_{2n+1}$  is found to contain more than one  $\Delta_n$ , and so to be separable in more than one way into two sets.

**THEOREM.** *If a  $\Delta_{15}$  contains two  $\Delta_7$ 's, they have in common one triad and no other letters. If a  $\Delta_{31}$  contains two  $\Delta_{15}$ 's, they have in common seven triads constituting a  $\Delta_7$ , but no other letters.*

*Proof.*—It is sufficient to demonstrate the second statement, for the same proof applies to the general theorem. Let the  $\Delta_{31}$  contain  $\Delta_{15}$  and  $\Delta'_{15}$ , two triad systems on 15's. Denote the set of 15 letters in  $\Delta_{15}$  by  $a, b, c, \dots$ , and

its even set of 16 by  $\alpha, \beta, \gamma, \dots, \pi$ . The  $\Delta'_{15}$ , as different from  $\Delta_{15}$ , must contain letters from both sets. Call one even letter in  $\Delta'_{15}$   $\pi$ . This letter  $\pi$  will occur in 7 triads of  $\Delta'_{15}$ , and since  $\pi$  is of the even set in  $\Delta_{31}$ , these triads are of the type  $\pi a \alpha$ . Consequently the  $\Delta'_{15}$  contains, beside  $\pi$ , seven even letters and seven odd. Any triad containing 2 of these 7 odd letters must contain a third odd letter; they constitute therefore a closed set of 7 triads, a  $\Delta_7$ , both in  $\Delta_{15}$  and in  $\Delta'_{15}$ . Further we note that the 8 even letters,  $\pi, \alpha, \dots$ , occurring in  $\Delta'_{15}$  are excluded from  $\Delta_{15}$ ; and the theorem is proved.

There are 8 residual letters of the even set, excluded both from  $\Delta_{15}$  and  $\Delta'_{15}$ . These must form, with each of the 7 odd letters common to both systems, 4 triads of the type  $a \alpha \beta$ ; in all, 28 triads. Combine the 7 common triads with these 28 external to both systems, and we see a third system,  $\Delta''_{15}$ , containing the 7 letters of the common set and the residual 8 even letters. We therefore state the theorem more fully thus:

**THEOREM.** *If a  $\Delta_{31}$  contains two  $\Delta_{15}$ 's, it contains also a third; and the three  $\Delta_{15}$ 's have in common a  $\Delta_7$ , but no letters except those involved in the  $\Delta_7$ . The three together involve all 31 letters, and contain 91 of the 155 triads that occur in the  $\Delta_{31}$ .*

**COROLLARY 1.** *If a  $\Delta_{31}$  contains a  $\Delta_{15}$  which contains no  $\Delta_7$  (a headless  $\Delta_{15}$ ), then the  $\Delta_{31}$  contains no other  $\Delta_{15}$ .*

**COROLLARY 2.** *If a  $\Delta_{31}$  contains a  $\Delta_{15}$  which contains only one  $\Delta_7$ , then it cannot contain more than two other  $\Delta_{15}$ 's.*

**COROLLARY 3.** *A  $\Delta_{31}$  which contains a  $\Delta_{15}$  of the kind showing only 3  $\Delta_7$ 's cannot have more than seven  $\Delta_{15}$ 's in all. The remaining possibilities are evidently fifteen and thirty-one.*

## 2. SYNTHESIS OF A $\Delta_{31}$ OF ODD AND EVEN STRUCTURE

It is the first of the preceding corollaries that I wish to employ here in proving non-equivalence. There is a well-known method, generalized from Reiss, of constructing a  $\Delta_{31}$  by the use of two  $\Delta_{15}$ 's. These may be either different or equivalent systems; the one is used in forming an array, 15 columns containing 8 pairs of letters each, the other is then superposed, in the form of 15 columns, each of which consists of 7 pairs with one letter at the head of the column. The letters of this latter system,  $a, b, c, \dots$ , are set as column-heads along a line, and in column below each one are written the 7 pairs or dyads that occur with it in triads of the system. To form the  $15 \times 8$  array, the first system, consisting of 15 other letters  $\alpha, \beta, \gamma, \dots$ , is written out in a similar tabular form, then with each letter at the head of a column is written a 16th letter, as  $x$ . This gives us all the 120 pairs that can be made from the 16 letters  $\alpha, \beta, \gamma, \dots, x$ , with no pair repeated. These 15 columns of 8, set below the other 15 columns of 7 in any order, will constitute a table of a

$\Delta_{31}$ , — the odd set of 15 above, the even set of 16 below. *If the first  $\Delta_{15}$ , used as here described, is of the headless variety (contains no  $\Delta_7$ ), the resulting  $\Delta_{31}$  will contain no other  $\Delta_{15}$ .* For we have then the case of Corollary 1, § 1.

As stated already, the order of columns in the lower array is entirely arbitrary, so that the number of possible juxtapositions when the upper system is superposed will be 15! But the number of non-equivalent  $\Delta_{31}$ 's resulting may be less. Any substitution on the letters  $a, b, c, \dots$  that transforms the superposed  $\Delta_{15}$  into itself will merely shift the relative positions of its columns; and similarly any substitution belonging to the group of the lower array, the  $\Delta_{31}$  being transformed in either case into an equivalent  $\Delta_{31}$  (by the definition of equivalence). Our proposed permutations of entire columns in the lower array constitute a group  $G_{15!}$ . If two resulting systems are equivalent, it must be through the equivalence of their unique  $\Delta_{15}$ 's in the upper array and the equivalence of their respective lower arrays. But the upper arrays are identical, whence their equivalence must subsist in a transformation of that upper array (i. e., of the headless  $\Delta_{15}$  which it tabulates) into itself. The group of that headless  $\Delta_{15}$  is assumed as given, so that we have now to inquire what subgroup of the  $G_{15!}$  on the lower columns is capable of being induced by substitutions on the 16 letters of that array? The problem of equivalence is now resolved into two problems of invariance, one solved in advance, the other more intricate, as will be seen.

### 3. THE GROUP OF A 15 BY 8 ARRAY DERIVED FROM A $\Delta_{15}$

The even set of a  $\Delta_{31}$ , when the system is separable into even and odd sets, can be displayed in fifteen columns, each containing eight pairs formed from letters of this even set, while above each column stands the letter of the odd set which completes its pairs to triads. Such an array is not always derivable from a  $\Delta_{15}$  by the method explained in section 2 above, and we do not propose here the general problem of its possible varieties. We require for present purposes only those that are so derivable from some triad system on 15 letters. Let the system contain a triad  $\alpha\beta\gamma$ . This will give rise to pairs in three columns, those headed by  $\alpha, \beta$ , and  $\gamma$  respectively. After a letter  $x$  has been combined with each such head, the three columns will contain two pairs each, related like the following:

$$\begin{array}{ccc} x\alpha & x\beta & x\gamma \\ \beta\gamma & \alpha\gamma & \alpha\beta. \end{array}$$

These exhibit what Cole terms triple interlacing,\* and Miss Cummings† styles

\* F. N. Cole, *The triad systems of 13 letters*. These Transactions, vol. 14 (1913), p. 1.

† L. D. Cummings. These Transactions, vol. 15 (1914), p. 311.

a complete quadrangle. As there are 35 triads in a  $\Delta_{15}$ , the particular letter  $x$  will participate in 35 such quadrangles; each pair  $x\alpha$  with all seven other pairs in its column will give rise to seven quadrangles.  $((15 \times 7) \div 3 = 35.)$  Conversely, of course, if in any such 15 by 8 array there is one letter  $x$  that occurs in 35 quadrangles, that array is derived from a  $\Delta_{15}$ , namely from the  $\Delta_{15}$  whose tabulation appears upon the removal of that letter  $x$  from the pair at the head of each column. This observation provides a test easily applied; for if two pairs in any column are seen not to be concerned in a complete quadrangle, none of the four letters involved can be the letter  $x$  of this discussion.

This test will reveal of course all the different  $\Delta_{15}$ 's that may be sources of any given 15 by 8 array. We apply it to arrays that have been formed from all  $\Delta_{15}$ 's now known (including headless  $\Delta_{15}$ 's not yet published), in order to learn whether any such array can have two different  $\Delta_{15}$ 's for its source. The result is negative in every case, with two exceptions, namely the systems III A and III B of Miss Cummings. *Any 15 by 8 array containing the 120 pairs that can be formed from 15 letters, no letter occurring twice in a column, if it is derivable from more than one  $\Delta_{15}$ , must come from either the Kirkman system, III A, or from the system III B of Miss Cummings's dissertation.*

These two exceptional cases we shall exclude from present consideration, and so simplify the question of the group of an array. Any substitution on its 16 letters which merely shifts the 15 columns must evidently also transform into itself any  $\Delta_{15}$  which can be a source, or else interchange among themselves two or more such sources. As we shall consider only those arrays that have a unique  $\Delta_{15}$  as source, we adopt always the first of these alternatives. *The group of the array is the same as that of the  $\Delta_{15}$  from which it is derived.* Or better, if we think of the group of an array as operating directly, not on its letters, but upon the numbers that denote the order of succession of its columns, we should say that the two groups are holoedrically isomorphic. But our question was this: How many permutations of the entire columns of the array are the same in effect as a mere interchange among the 16 letters? This question too has been answered by implication when we answered its converse.

Let us recapitulate, since this is the essential point in our proof. We have a superposed  $\Delta_{15}$ , the choice of which insures its uniqueness in the  $\Delta_{31}$ , arranged in 15 columns. Below these we have the 15 columns of an array, 15 by 8, derivable from only one  $\Delta_{15}$ . The columns of this array are permuted in  $15!$  ways; and it is found that equivalent  $\Delta_{31}$ 's are produced only when one permutation is convertible into the other either by (1) an operation on the superposed partial columns belonging in the group of their  $\Delta_{15}$ , or

by (2) an operation in the group of the  $\Delta_{15}$  from which the lower array of partial columns is derived, or by a combination of both. The orders of these two groups being denoted by  $d$  and  $d'$  respectively, we now remark further that the number of operations common to the two groups (call it  $r$ ), will be the order of the group of the resulting  $\Delta_{31}$ , and that the product  $dd'$  divided by  $r$  will give the number of systems  $\Delta_{31}$  equivalent to any one. Of course  $r$  need not be the same for two non-equivalent sets of arrangements, but no such set of equivalent arrangements can number more than  $dd'$ . For forming an approximate idea, this number  $dd'$  will be used, although it is often too large. *Then for any one superposed  $\Delta_{15}$  and any one subterposed array, the number of non-equivalent  $\Delta_{31}$ 's produced cannot be less than  $15!/(dd')$ .*

#### 4. COMPOUND TRIAD SYSTEMS ON 31 LETTERS, WHOSE ODD SET CONSTITUTE A HEADLESS $\Delta_{15}$

It is the intention to prove that the  $\Delta_{31}$ 's are exceedingly numerous, not to find their number with any precision. It will suffice if we examine roughly triad systems,  $\Delta_{31}$ 's produced as above described, which are compounded in sets of not more than 15 equivalent systems. This is an arbitrary bound, and we shall further reckon as if 15 were the precise number of conjugate, equivalent systems in each set. I take from Miss Cummings's dissertation (loc. cit.), and from my own unpublished work\* already referred to, data upon  $\Delta_{15}$ 's whose groups are of low order. A headless  $\Delta_{15}$  whose group is of order  $d$  will be used as head (i. e., odd set) for a  $\Delta_{31}$ ; and to furnish the even set, the 15 by 8 array, any  $\Delta_{15}$ , either the same or different, headless or itself compound, will be employed. We shall avoid, for the reasons stated, the Cummings systems III A and III B, so that never will the same array come from two sources and never will equivalent  $\Delta_{31}$ 's be counted as distinct. The table below is arranged for values of  $d$  and  $d'$ , showing in how many distinct  $\Delta_{15}$ 's each occurs. Of course then the  $d'$  relates to both classes of  $\Delta_{15}$ 's, the  $d$  to headless systems only.

Values of $d$	Values of $d'$	Resulting divisors, $dd'$ , of $15!$
2, 4 systems	2, 7 systems	4, 28 combinations
3, 8 "	3, 12 "	6, 104 "
4, 4 "	4, 7 "	8, 56 "
5, 1 system	5, 1 system	9, 96 "
6, 1 "	6, 1 "	10, 11 "
	8, 2 systems	12, 115 "
		15, 20 "
		TOTAL, 430 divisors not above 15.

\* This research has later been communicated to the American Mathematical Society, October 31, 1914.

If these were all, and if these all arose in sets of 15 conjugate equivalent systems, even then their number would be 430 times  $15!/15$ , i. e. 430 (14!), which is above  $3 \times 10^{13}$ .

*Certainly therefore there exist more than thirty-seven million millions of triad systems on 31 letters, no two systems equivalent, of that restricted kind which contain one  $\Delta_{15}$ , which itself contains no  $\Delta_7$ .*

This leaves untouched many questions as to other kind of arrays and other kinds of  $\Delta_{31}$ 's, and particularly any kind that may contain no  $\Delta_{15}$ .

Similar reasoning carried out for triad systems on 27 letters, constructed from the two kinds on 13 letters whose groups are of orders 6 and 39 respectively, shows a number greater than 222 millions.

Let me call attention also to the fact that  $\Delta_{31}$ 's actually exist whose group is the identity only. For if  $d$  and  $d'$  are relatively prime, the subgroup which forms the group of the  $\Delta_{31}$  must be of order 1. This kind of triad systems is found therefore in great number for 31 letters, but none have yet been announced for a less number of letters.

VASSAR COLLEGE,  
August, 1914.

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